

LOWER BOUND ESTIMATES FOR EIGENVALUES OF THE LAPLACIAN*

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ABSTRACT. For an n -dimensional polytope Ω in \mathbb{R}^n , we study lower bounds for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian. In the asymptotic formula on the average of the first k eigenvalues, Li and Yau [4] obtained the first term with the order $k^{\frac{2}{n}}$, which is optimal. The next landmark goal is to give the second term with the order $k^{\frac{1}{n}}$ in the asymptotic formula. For this purpose, Kovařík, Vugalter and Weidl [3] have made an important breakthrough in the case of dimension 2. It is our purpose to study the n -dimensional case for arbitrary dimension n . We obtain the second term in the asymptotic sense.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary $\partial\Omega$ in an n -dimensional Euclidean space \mathbb{R}^n , $n \geq 2$. We consider the following Dirichlet eigenvalue problem of the Laplacian:

$$(1.1) \quad \begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that the spectrum of this problem is real and discrete:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \longrightarrow +\infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

Let $V(\Omega)$ denote the volume of Ω and let B_n denote the volume of the unit ball in \mathbb{R}^n . One has the following Weyl's asymptotic formula

$$(1.2) \quad \lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \rightarrow +\infty.$$

From the above asymptotic formula, one can obtain

$$(1.3) \quad \frac{1}{k} \sum_{j=1}^k \lambda_j \sim \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \rightarrow +\infty.$$

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Furthermore, Pólya [7] proved that

$$(1.4) \quad \lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots,$$

if Ω is a tiling domain in \mathbb{R}^n . Moreover, he proposed the following

Conjecture of Pólya. *If Ω is a bounded domain in \mathbb{R}^n , then the k -th eigenvalue λ_k of the eigenvalue problem (1.1) satisfies*

$$(1.5) \quad \lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots$$

On the conjecture of Pólya, much work has been done ([1], [4], [5]). In particular, Li and Yau [4] proved the following

$$(1.6) \quad \frac{1}{k} \sum_{j=1}^k \lambda_j \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots$$

The formula (1.3) shows that the constant in the result (1.6) of Li and Yau can not be improved. From this formula (1.6), one can derive

$$(1.7) \quad \lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots,$$

which gives a partial solution for the conjecture of Pólya with a factor $\frac{n}{n+2}$. Recently, Melas [6] improved the estimate (1.6) to the following:

$$(1.8) \quad \frac{1}{k} \sum_{j=1}^k \lambda_j \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + M_n \frac{V(\Omega)}{I(\Omega)}, \quad \text{for } k = 1, 2, \dots,$$

where M_n is a positive constant depending only on the dimension n and

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 dx$$

is called *the moment of inertia* of Ω .

For the average of the first k eigenvalues, it is important to compare its lower bound with the following asymptotical behavior:

$$(1.9) \quad \frac{1}{k} \sum_{j=1}^k \lambda_j = \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + C_n \frac{A(\partial\Omega)}{V(\Omega)^{1+\frac{1}{n}}} k^{\frac{1}{n}} + o(k^{\frac{1}{n}}), \quad k \rightarrow +\infty,$$

where $A(\partial\Omega)$ denotes the $(n-1)$ -dimensional volume of $\partial\Omega$ and C_n is a positive constant depending only on the dimension n . The first term in (1.9) is due to Weyl [9]. In [8], the second term in (1.9) was established under suitable conditions on Ω . Since the first asymptotical term is optimal, the next landmark goal on its lower bound estimate is to obtain the second asymptotical term with the order of $k^{\frac{1}{n}}$. For this purpose, Kovařík, Vugalter and Weidl [3] have made an important breakthrough for this landmark goal in the case of dimension 2. They have added a positive term in the right hand side of (1.6), which is similar to the second term of (1.9) in the asymptotic sense. The purpose of this paper is to study the n -dimensional case for arbitrary dimension n . We also obtain

the second term of (1.9) in the asymptotic sense. For estimates on upper bounds of eigenvalues, one can see Cheng and Yang [2].

For an n -dimensional polytope Ω in \mathbb{R}^n , we denote by p_i , $i = 1, \dots, m$, the i -th face of Ω . Assume that A_i is the area of the i -th face p_i of Ω . For each $i = 1, \dots, m$, we choose several non-overlapping $(n-1)$ -dimensional convex subdomains s_{r_i} in the interior of p_i such that the area of $\bigcup s_{r_i}$ is greater than or equal to one third of A_i and the distance d_i between $\bigcup s_{r_i}$ and $\partial\Omega \setminus p_i$ is greater than 0. Define the function $\Theta : \mathbb{R} \rightarrow \mathbb{R}$ by $\Theta(t) = 0$ if $t \leq 0$ and $\Theta(t) = 1$ if $t > 0$. Then we prove the following

Theorem 1. *Let Ω be an n -dimensional polytope in \mathbb{R}^n . Then, for any positive integer k , we have*

$$\frac{1}{k} \sum_{j=1}^k \lambda_j \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{3^{-4} 2^{3-n} \pi^2}{(n+2) B_n^{\frac{2}{n}}} \frac{A(\partial\Omega)}{V(\Omega)^{1+\frac{2}{n}}} \left(\frac{V(\Omega) \lambda_k}{\alpha_1} \right)^{-n\varepsilon(k)} k^{\frac{2}{n}} \lambda_k^{-\frac{1}{2}} \Theta(\lambda_k - \lambda_0),$$

where

$$\varepsilon(k) = \left[\sqrt{\frac{\log_2((V(\Omega)/\alpha_1)^{n-1} \lambda_k^{\frac{n}{2}})}{n+12}} \right]^{-1}, \quad \alpha_1 = \sqrt{\frac{3}{B_n} \left(\frac{4n\pi^2}{n+2} \right)^{\frac{n}{2}}},$$

$$\lambda_0 = \max \left\{ \frac{4n}{\min_i \{d_i^2\}}, \left(\frac{\alpha_1}{V(\Omega)} \right)^{\frac{2}{n}}, 2^{\frac{2(n+12)}{n}} \left(\frac{\alpha_1}{V(\Omega)} \right)^{\frac{2(n-1)}{n}}, \left(\frac{12}{\min_i \{A_i\}} \right)^{\frac{2}{n-1}} \right\}.$$

Remark 1. Notice that $\varepsilon(k) \rightarrow 0$ and $\lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}$ as $k \rightarrow +\infty$. It shows that the second term on the right hand side of the inequality in Theorem 1 is very similar to the second term in the asymptotic (1.9) when k is large enough. Combining this with the inequality (1.7), we immediately obtain the following

Corollary 1. *Let Ω be an n -dimensional polytope in \mathbb{R}^n . Then, there exists a positive integer N , such that, for all $k \geq N$,*

$$\frac{1}{k} \sum_{j=1}^k \lambda_j \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{\pi}{3^4 2^{n-1} (n+2) B_n^{\frac{1}{n}}} \frac{A(\partial\Omega)}{V(\Omega)^{1+\frac{1}{n}}} k^{\frac{1}{n}-2\varepsilon(k)},$$

where

$$\varepsilon(k) = \left[\sqrt{\frac{1}{n+12} \log_2 \left(\left(\frac{V(\Omega)}{\alpha_1} \right)^{n-1} \left(\frac{4n\pi^2}{n+2} \right)^{\frac{n}{2}} \frac{k}{B_n V(\Omega)} \right)} \right]^{-1}, \quad \alpha_1 = \sqrt{\frac{3}{B_n} \left(\frac{4n\pi^2}{n+2} \right)^{\frac{n}{2}}}.$$

2. PROOF OF MAIN THEOREM

For an n -dimensional polytope Ω in \mathbb{R}^n , let u_j be a normalized eigenfunction corresponding to the j -th eigenvalue λ_j , i.e. u_j satisfies

$$(2.1) \quad \begin{cases} \Delta u_j = -\lambda_j u_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} u_j u_k = \delta_{jk}, & \forall j, k. \end{cases}$$

Then $\{u_j\}_{j=1}^{+\infty}$ forms an orthonormal basis of $L^2(\Omega)$. We consider the function φ_j given by

$$\varphi_j(x) = \begin{cases} u_j(x) & , \quad x \in \Omega, \\ 0 & , \quad x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Denote by $\widehat{\varphi}_j$ the Fourier transform of φ_j . For any $\xi \in \mathbb{R}^n$, we have

$$\widehat{\varphi}_j(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi_j(x) e^{\sqrt{-1}\langle \xi, x \rangle} dx = (2\pi)^{-\frac{n}{2}} \int_{\Omega} u_j(x) e^{\sqrt{-1}\langle \xi, x \rangle} dx.$$

Take λ large enough such that λ satisfies the following two conditions:

- (i) for each $i = 1, \dots, m$, there exist some non-overlapping $(n-1)$ -dimensional cubes t_{l_i} with the side $\frac{1}{\sqrt{\lambda}}$ on $\bigcup s_{r_i}$, whose total area is greater than or equal to $\frac{1}{6}A_i$;
- (ii) $\frac{1}{\sqrt{\lambda}} \leq \frac{1}{2\sqrt{n}} \min_i \{d_i\} \leq \frac{d_i}{2\sqrt{n}}$. In this case, we make sure that the n -dimensional rectangles $T_{l_i} = [0, \frac{1}{2\sqrt{\lambda}}] \times t_{l_i}$ lie inside Ω and they do not overlap each other.

We define a function F_{λ} by

$$F_{\lambda}(\xi) = \sum_{\lambda_j \leq \lambda} |\widehat{\varphi}_j(\xi)|^2.$$

By Parseval's identity, we have

$$(2.2) \quad \int_{\mathbb{R}^n} F_{\lambda}(\xi) d\xi = \sum_{\lambda_j \leq \lambda} \int_{\mathbb{R}^n} |\widehat{\varphi}_j(\xi)|^2 d\xi = \sum_{\lambda_j \leq \lambda} \int_{\mathbb{R}^n} \varphi_j^2(x) dx = \sum_{\lambda_j \leq \lambda} \int_{\Omega} u_j^2(x) dx = N(\lambda),$$

where $N(\lambda)$ is the number of eigenvalues $\lambda_j \leq \lambda$. Furthermore, we deduce from integration by parts and Parseval's identity that

$$(2.3) \quad \int_{\mathbb{R}^n} |\xi|^2 F_{\lambda}(\xi) d\xi = \sum_{\lambda_j \leq \lambda} \int_{\mathbb{R}^n} |\xi|^2 |\widehat{\varphi}_j(\xi)|^2 d\xi = - \sum_{\lambda_j \leq \lambda} \int_{\Omega} u_j \Delta u_j dx = \sum_{\lambda_j \leq \lambda} \lambda_j.$$

For each fixed $\xi \in \mathbb{R}^n$, since $e^{\sqrt{-1}\langle \xi, x \rangle}$ belongs to $L^2(\Omega)$, it follows that

$$e^{\sqrt{-1}\langle \xi, x \rangle} = \sum_{j=1}^{\infty} c_j(\xi) u_j, \quad \text{where } c_j(\xi) = \int_{\Omega} u_j(x) e^{\sqrt{-1}\langle \xi, x \rangle} dx.$$

Let

$$u(\xi, x) = \sum_{\lambda_j \leq \lambda} c_j(\xi) u_j(x).$$

Then we have

$$(2.4) \quad \left\| u - e^{\sqrt{-1}\langle \xi, x \rangle} \right\|_{L^2(\Omega)}^2 = V - (2\pi)^n F_\lambda(\xi).$$

To prove Theorem 1, we will need the following lemma.

Lemma 2.1. ([4]) *If F is a real-valued function defined on \mathbb{R}^n with $0 \leq F \leq M_1$, and*

$$\int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi \leq M_2,$$

then

$$\int_{\mathbb{R}^n} F(\xi) d\xi \leq \left(\frac{n+2}{n} \right)^{\frac{n}{n+2}} M_2^{\frac{n}{n+2}} (M_1 B_n)^{\frac{2}{n+2}}.$$

Next we need to establish the estimate for $F_\lambda(\xi)$. For each l_i , we choose a local coordinate system (x_1, \dots, x_n) such that $t_{l_i} = [-\frac{1}{2\sqrt{\lambda}}, \frac{1}{2\sqrt{\lambda}}]^{n-1}$ and $\frac{\partial}{\partial x_1}$ is the inward unit normal vector field on t_{l_i} . To derive the upper bound of $F_\lambda(\xi)$, we prepare the following lemmas:

Lemma 2.2. *For any positive integer p , we have*

$$\left\| \frac{\partial^p u}{\partial x_1^p} \right\|_{L^2(T_{l_i})}^2 \leq \left(\frac{n+2}{4n\pi^2} \right)^{\frac{n}{2}} B_n V^2 D_{p-1} \lambda^{p+\frac{n}{2}},$$

where the sequence D_q is defined by

$$\begin{aligned} D_0 &= 1, & D_1 &= 3(1 + 44^2 n^2 p^4 + 4 \cdot 5^2 n p^2), \\ D_q &= 3(1 + 44^2 n^2 p^4) D_{q-2} + (12 \cdot 5^2 n p^2) D_{q-1}, & q &= 2, 3, \dots \end{aligned}$$

Proof. For $p \geq 1$ and $0 \leq q \leq p-1$, we define functions g and $v_{q,p}$ by the following

$$\begin{aligned} g(x) &= 1 - 6x^4 + 8x^6 - 3x^8, \quad 0 \leq x \leq 1, \\ v_{q,p}(t) &= \begin{cases} 1 & , \quad 0 \leq t \leq \frac{2p-q}{2p}, \\ g(2pt - 2p + q) & , \quad \frac{2p-q}{2p} \leq t \leq \frac{2p-q+1}{2p}, \\ 0 & , \quad \frac{2p-q+1}{2p} < t, \end{cases} \end{aligned}$$

with $v_{q,p}(-t) = v_{q,p}(t)$ for $t < 0$. From the definition of g , it follows that

$$(2.5) \quad |g(x)| \leq 1, \quad |g'(x)| < \frac{5}{2}, \quad |g''(x)| < 11.$$

By the definition of $v_{q,p}$ and (2.5), we get

$$(2.6) \quad |v_{q,p}(t)| \leq 1, \quad |v'_{q,p}(t)| < 5p, \quad |v''_{q,p}(t)| < 44p^2.$$

Next we define

$$W_{q,p,\lambda}(x_1, \dots, x_n) = v_{q,p}(\sqrt{\lambda} x_1) v_{q,p}(\sqrt{\lambda} x_2) \cdots v_{q,p}(\sqrt{\lambda} x_n), \quad (x_1, \dots, x_n) \in \mathbb{R}^n,$$

and set

$$\omega_q = \text{supp} W_{q,p,\lambda}.$$

Then we have

$$(2.7) \quad |W_{q,p,\lambda}| \leq 1, \quad |\nabla W_{q,p,\lambda}| < 5n^{\frac{1}{2}}\lambda^{\frac{1}{2}}p, \quad |\Delta W_{q,p,\lambda}| < 44n\lambda p^2,$$

and

$$(2.8) \quad T_{l_i} \subset \omega_q, \quad \omega_{q+1} \subset \omega_q, \quad W_{q,p,\lambda} \equiv 1 \text{ on } \omega_{q+1}, \quad \nabla W_{q,p,\lambda} = 0 \text{ on } \partial\omega_q.$$

We will prove

$$(2.9) \quad \left\| \nabla \left(\frac{\partial^q \varphi_j}{\partial x_1^q} \right) \right\|_{L^2(\omega_q)}^2 \leq D_q \lambda^{q+1},$$

by induction on q for $q = 0, 1, \dots, p-1$. For $q = 0$, since $D_0 = 1$, it follows that

$$(2.10) \quad \|\nabla \varphi_j\|_{L^2(\omega_0)}^2 \leq \int_{\Omega} |\nabla u_j|^2 = - \int_{\Omega} u_j \Delta u_j = \lambda_j \leq \lambda = D_0 \lambda.$$

Using the property (2.8) of $W_{q,p,\lambda}$ and integration by parts, we get

$$(2.11) \quad \begin{aligned} & \left\| \Delta(\varphi_j W_{0,p,\lambda}) \right\|_{L^2(\omega_0)}^2 \\ &= \sum_{k=1}^n \left\| \frac{\partial^2}{\partial x_k^2}(\varphi_j W_{0,p,\lambda}) \right\|_{L^2(\omega_0)}^2 + 2 \sum_{k < l} \int_{\omega_0} \frac{\partial^2}{\partial x_k^2}(\varphi_j W_{0,p,\lambda}) \frac{\partial^2}{\partial x_l^2}(\varphi_j W_{0,p,\lambda}) \\ &= \sum_{k=1}^n \left\| \frac{\partial^2}{\partial x_k^2}(\varphi_j W_{0,p,\lambda}) \right\|_{L^2(\omega_0)}^2 + 2 \sum_{k < l} \left\| \frac{\partial^2}{\partial x_k \partial x_l}(\varphi_j W_{0,p,\lambda}) \right\|_{L^2(\omega_0)}^2. \end{aligned}$$

From (2.7), (2.8), (2.10) and (2.11), it follows that

$$\begin{aligned} \left\| \nabla \left(\frac{\partial \varphi_j}{\partial x_1} \right) \right\|_{L^2(\omega_1)}^2 &= \sum_{k=1}^n \left\| \frac{\partial^2}{\partial x_1 \partial x_k}(\varphi_j W_{0,p,\lambda}) \right\|_{L^2(\omega_1)}^2 \\ &\leq \sum_{k=1}^n \left\| \frac{\partial^2}{\partial x_1 \partial x_k}(\varphi_j W_{0,p,\lambda}) \right\|_{L^2(\omega_0)}^2 \\ &\leq \left\| \Delta(\varphi_j W_{0,p,\lambda}) \right\|_{L^2(\omega_0)}^2 \\ &= \left\| -\lambda_j \varphi_j W_{0,p,\lambda} + \varphi_j \Delta W_{0,p,\lambda} + 2 \nabla \varphi_j \nabla W_{0,p,\lambda} \right\|_{L^2(\omega_0)}^2 \\ &\leq 3 \left\{ \lambda^2 (1 + 44^2 n^2 p^4) \|\varphi_j\|_{L^2(\omega_0)}^2 + 4 \cdot 5^2 p^2 n \lambda \|\nabla \varphi_j\|_{L^2(\omega_0)}^2 \right\} \\ &\leq 3 \left\{ \lambda^2 (1 + 44^2 n^2 p^4) + 4 \cdot 5^2 p^2 n \lambda^2 \right\} = D_1 \lambda^2. \end{aligned}$$

Hence (2.9) holds for $q = 0$ and $q = 1$. Now assume that (2.9) holds some $q - 1$ and q . We will show that it holds for $q + 1$ as well. Notice that

$$W_{q,p,\lambda} = \nabla W_{q,p,\lambda} = 0 \quad \text{on } \partial\omega_q,$$

then

$$(2.12) \quad \begin{aligned} & \left\| \Delta \left(\frac{\partial^q \varphi_j}{\partial x_1^q} W_{q,p,\lambda} \right) \right\|_{L^2(\omega_q)}^2 \\ &= \sum_{k=1}^n \left\| \frac{\partial^2}{\partial x_k^2} \left(\frac{\partial^q \varphi_j}{\partial x_1^q} W_{q,p,\lambda} \right) \right\|_{L^2(\omega_q)}^2 + 2 \sum_{k < l} \left\| \frac{\partial^2}{\partial x_k \partial x_l} \left(\frac{\partial^q \varphi_j}{\partial x_1^q} W_{q,p,\lambda} \right) \right\|_{L^2(\omega_q)}^2. \end{aligned}$$

From (2.7), (2.8) and (2.12), it follows that

$$\begin{aligned} & \left\| \nabla \left(\frac{\partial^{q+1} \varphi_j}{\partial x_1^{q+1}} \right) \right\|_{L^2(\omega_{q+1})}^2 \\ &= \sum_{k=1}^n \left\| \frac{\partial^2}{\partial x_1 \partial x_k} \left(\frac{\partial^q \varphi_j}{\partial x_1^q} W_{q,p,\lambda} \right) \right\|_{L^2(\omega_{q+1})}^2 \\ &\leq \sum_{k=1}^n \left\| \frac{\partial^2}{\partial x_1 \partial x_k} \left(\frac{\partial^q \varphi_j}{\partial x_1^q} W_{q,p,\lambda} \right) \right\|_{L^2(\omega_q)}^2 \\ &\leq \left\| \Delta \left(\frac{\partial^q \varphi_j}{\partial x_1^q} W_{q,p,\lambda} \right) \right\|_{L^2(\omega_q)}^2 \\ &= \left\| -\lambda_j \left(\frac{\partial^q \varphi_j}{\partial x_1^q} \right) W_{q,p,\lambda} + \left(\frac{\partial^q \varphi_j}{\partial x_1^q} \right) \Delta W_{q,p,\lambda} + 2 \nabla \left(\frac{\partial^q \varphi_j}{\partial x_1^q} \right) \nabla W_{q,p,\lambda} \right\|_{L^2(\omega_q)}^2 \\ &\leq 3 \left\{ \lambda^2 (1 + 44^2 n^2 p^4) \left\| \frac{\partial^q \varphi_j}{\partial x_1^q} \right\|_{L^2(\omega_q)}^2 + 4 \cdot 5^2 p^2 n \lambda \left\| \nabla \frac{\partial^q \varphi_j}{\partial x_1^q} \right\|_{L^2(\omega_q)}^2 \right\} \\ &\leq 3 \left\{ \lambda^2 (1 + 44^2 n^2 p^4) \left\| \nabla \left(\frac{\partial^{q-1} \varphi_j}{\partial x_1^{q-1}} \right) \right\|_{L^2(\omega_{q-1})}^2 + 4 \cdot 5^2 p^2 n \lambda \left\| \nabla \frac{\partial^q \varphi_j}{\partial x_1^q} \right\|_{L^2(\omega_q)}^2 \right\} \\ &\leq \lambda^{q+2} \left\{ 3(1 + 44^2 n^2 p^4) D_{q-1} + 12 \cdot 5^2 n p^2 D_q \right\} = D_{q+1} \lambda^{q+2}. \end{aligned}$$

This implies that (2.9) holds for $q + 1$. Therefore, (2.9) holds for any integer $0 \leq q \leq p - 1$. Taking $q = p - 1$ in (2.9), we can obtain

$$(2.13) \quad \left\| \frac{\partial^p u_j}{\partial x_1^p} \right\|_{L^2(T_{l_i})}^2 \leq D_{p-1} \lambda^p.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left\| \frac{\partial^p u}{\partial x_1^p} \right\|_{L^2(T_{l_i})}^2 &= \left\| \sum_{\lambda_j \leq \lambda} c_j \frac{\partial^p u_j}{\partial x_1^p} \right\|_{L^2(T_{l_i})}^2 = \int_{T_{l_i}} \left| \sum_{\lambda_j \leq \lambda} c_j \frac{\partial^p u_j}{\partial x_1^p} \right|^2 \\ &\leq \int_{T_{l_i}} \sum_{\lambda_j \leq \lambda} |c_j|^2 \sum_{\lambda_j \leq \lambda} \left| \frac{\partial^p u_j}{\partial x_1^p} \right|^2 \leq V \sum_{\lambda_j \leq \lambda} \left\| \frac{\partial^p u_j}{\partial x_1^p} \right\|_{L^2(T_{l_i})}^2 \leq V \cdot N(\lambda) D_{p-1} \lambda^p. \end{aligned}$$

Using the lower bound of λ_j given in (1.7), we find out that

$$N(\lambda) \leq B_n V \left(\frac{n+2}{4n\pi^2} \right)^{\frac{n}{2}} \lambda^{\frac{n}{2}}.$$

Then

$$\left\| \frac{\partial^p u}{\partial x_1^p} \right\|_{L^2(T_{l_i})}^2 \leq \left(\frac{n+2}{4n\pi^2} \right)^{\frac{n}{2}} B_n V^2 D_{p-1} \lambda^{p+\frac{n}{2}}. \quad \square$$

Lemma 2.3. *Let p be a positive integer and let $f \in C^p[0, \frac{1}{2\sqrt{\lambda}}]$ be a real-valued function. If $f^{(p)}$ is not identically zero, then one of the following inequalities holds true:*

$$\begin{aligned} \max|f'| &\leq 2^{p-1} (\max|f^{(p)}|)^{\frac{1}{p}} (\max|f|)^{1-\frac{1}{p}}, \\ \max|f'| &< 4^{p+1} \lambda^{\frac{1}{2}} \max|f|. \end{aligned}$$

Proof. Let $m_q = \max|f^{(q)}|$, $q \in \{0, 1, \dots, p\}$. If $p = 1$, then the conclusion is obvious. Next we assume that $p \geq 2$. For any fixed $0 \leq q \leq p-2$, we let $m_{q+1} = |f^{(q+1)}(t_0)|$, $t_0 \in [0, \frac{1}{2\sqrt{\lambda}}]$. We discuss separately the following cases:

Case 1: $t_0 < \frac{1}{4\sqrt{\lambda}}$. In this case, if $\frac{m_{q+1}}{m_{q+2}} \leq \frac{1}{4\sqrt{\lambda}}$, then it follows from Taylor's formula that

$$\begin{aligned} m_q &\geq \left| f^{(q)} \left(t_0 + \frac{m_{q+1}}{m_{q+2}} \right) \right| = \left| f^{(q)}(t_0) + f^{(q+1)}(t_0) \cdot \frac{m_{q+1}}{m_{q+2}} + \frac{1}{2} f^{(q+2)}(\xi) \left(\frac{m_{q+1}}{m_{q+2}} \right)^2 \right| \\ &\geq \left| f^{(q+1)}(t_0) \cdot \frac{m_{q+1}}{m_{q+2}} \right| - \left| f^{(q)}(t_0) \right| - \left| \frac{1}{2} f^{(q+2)}(\xi) \left(\frac{m_{q+1}}{m_{q+2}} \right)^2 \right| \\ &\geq -m_q + \frac{m_{q+1}^2}{m_{q+2}} - \frac{m_{q+2}}{2} \left(\frac{m_{q+1}}{m_{q+2}} \right)^2 \\ &= -m_q + \frac{m_{q+1}^2}{2m_{q+2}}, \end{aligned}$$

which implies

$$(2.14) \quad \frac{m_{q+2}}{m_{q+1}} \geq \frac{1}{4} \frac{m_{q+1}}{m_q}.$$

If $\frac{m_{q+1}}{m_{q+2}} > \frac{1}{4\sqrt{\lambda}}$, then we have

$$\begin{aligned} m_q &\geq \left| f^{(q)} \left(t_0 + \frac{1}{4\sqrt{\lambda}} \right) \right| = \left| f^{(q)}(t_0) + f^{(q+1)}(t_0) \cdot \frac{1}{4\sqrt{\lambda}} + \frac{1}{2} f^{(q+2)}(\xi) \left(\frac{1}{4\sqrt{\lambda}} \right)^2 \right| \\ &\geq -m_q + \frac{m_{q+1}}{4\sqrt{\lambda}} - \frac{m_{q+2}}{2} \left(\frac{1}{4\sqrt{\lambda}} \right)^2 \\ &> -m_q + \frac{m_{q+1}}{4\sqrt{\lambda}} - \frac{4\sqrt{\lambda} m_{q+1}}{2} \left(\frac{1}{4\sqrt{\lambda}} \right)^2, \end{aligned}$$

which implies

$$(2.15) \quad \frac{m_{q+1}}{m_q} < 16\sqrt{\lambda}.$$

Case 2: $t_0 \geq \frac{1}{4\sqrt{\lambda}}$. In this case, if $\frac{m_{q+1}}{m_{q+2}} \leq \frac{1}{4\sqrt{\lambda}}$, then we have

$$\begin{aligned} m_q &\geq \left| f^{(q)}(t_0 - \frac{m_{q+1}}{m_{q+2}}) \right| = \left| f^{(q)}(t_0) + f^{(q+1)}(t_0) \left(-\frac{m_{q+1}}{m_{q+2}} \right) + \frac{1}{2} f^{(q+2)}(\xi) \left(-\frac{m_{q+1}}{m_{q+2}} \right)^2 \right| \\ &\geq -m_q + \frac{m_{q+1}^2}{m_{q+2}} - \frac{m_{q+2}}{2} \left(\frac{m_{q+1}}{m_{q+2}} \right)^2 \\ &= -m_q + \frac{m_{q+1}^2}{2m_{q+2}}, \end{aligned}$$

which implies (2.14). If $\frac{m_{q+1}}{m_{q+2}} > \frac{1}{4\sqrt{\lambda}}$, then we have

$$\begin{aligned} m_q &\geq \left| f^{(q)}(t_0 - \frac{1}{4\sqrt{\lambda}}) \right| = \left| f^{(q)}(t_0) + f^{(q+1)}(t_0) \left(-\frac{1}{4\sqrt{\lambda}} \right) + \frac{1}{2} f^{(q+2)}(\xi) \left(-\frac{1}{4\sqrt{\lambda}} \right)^2 \right| \\ &\geq -m_q + \frac{m_{q+1}}{4\sqrt{\lambda}} - \frac{m_{q+2}}{2} \left(\frac{1}{4\sqrt{\lambda}} \right)^2 \\ &> -m_q + \frac{m_{q+1}}{4\sqrt{\lambda}} - \frac{4\sqrt{\lambda} m_{q+1}}{2} \left(\frac{1}{4\sqrt{\lambda}} \right)^2, \end{aligned}$$

which implies (2.15). Therefore, for any fixed $0 \leq q \leq p-2$, one of the inequalities (2.14) and (2.15) holds true.

Meanwhile, we note that there are two possibilities. Either for all $0 \leq q \leq p-1$,

$$(2.16) \quad \frac{m_{q+1}}{m_q} \geq 16\sqrt{\lambda},$$

or there exists $q_0 \in \{0, 1, \dots, p-1\}$, such that

$$(2.17) \quad \forall 0 \leq q < q_0, \quad \frac{m_{q+1}}{m_q} \geq 16\sqrt{\lambda}, \quad \frac{m_{q_0+1}}{m_{q_0}} < 16\sqrt{\lambda}.$$

If (2.16) holds, then we apply (2.14) to get

$$\begin{aligned} \frac{m_p}{m_0} &= \frac{m_p}{m_{p-1}} \cdot \frac{m_{p-1}}{m_{p-2}} \cdots \frac{m_2}{m_1} \cdot \frac{m_1}{m_0} \\ &\geq \frac{1}{4} \frac{m_{p-1}}{m_{p-2}} \cdot \frac{1}{4} \frac{m_{p-2}}{m_{p-3}} \cdots \frac{1}{4} \frac{m_1}{m_0} \cdot \frac{m_1}{m_0} \\ &= \left(\frac{1}{4} \right)^{p-1} \cdot \frac{m_{p-1}}{m_0} \cdot \frac{m_1}{m_0}, \end{aligned}$$

which implies

$$\frac{m_p}{m_0} \geq \left(\frac{1}{4} \right)^{(p-1)+\dots+1} \left(\frac{m_1}{m_0} \right)^p = 4^{-\frac{p(p-1)}{2}} \left(\frac{m_1}{m_0} \right)^p.$$

Thus,

$$\frac{m_1}{m_0} \leq 2^{p-1} \left(\frac{m_p}{m_0} \right)^{\frac{1}{p}}.$$

If (2.17) holds, we apply (2.14) to obtain

$$\frac{m_1}{m_0} \leq 4 \frac{m_2}{m_1} \leq \dots \leq 4^{q_0} \frac{m_{q_0+1}}{m_{q_0}} < 4^{q_0} \cdot 16\sqrt{\lambda} = 4^{q_0+2} \lambda^{\frac{1}{2}} \leq 4^{p+1} \lambda^{\frac{1}{2}}.$$

This completes the proof. \square

Lemma 2.4. *Let p be a positive integer. Then for any $\xi \in \mathbb{R}^n$,*

$$\left\| u - e^{\sqrt{-1}\langle \xi, x \rangle} \right\|_{L^2(T_{i_i})}^2 \geq \frac{1}{9 \cdot 2^{n-1}} \min \left\{ 2^{-2p-5} \lambda^{-\frac{n}{2}}, \quad 2^{-p-2} 6^{\frac{1}{2p}} (\beta_p^2 + \beta_{p+1}^2)^{-\frac{1}{2p}} \lambda^{-\frac{n}{2} - \frac{n}{2p}} \right\},$$

where

$$\beta_q^2 = \left(\frac{n+2}{4n\pi^2} \right)^{\frac{n}{2}} B_n V^2 D_{q-1}, \quad q \in \{p, p+1\}.$$

Proof. By Lemma 2.2, we have

$$\left\| \frac{\partial^q u}{\partial x_1^q} \right\|_{L^2(T_{i_i})}^2 \leq \beta_q^2 \lambda^{q+\frac{n}{2}}.$$

This implies that the measure of the set

$$\left\{ (x_2, \dots, x_n) \in \left[-\frac{1}{2\sqrt{\lambda}}, \frac{1}{2\sqrt{\lambda}} \right]^{n-1} : \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \frac{\partial^q u}{\partial x_1^q} \right|^2 dx_1 \leq 2\beta_q^2 \lambda^{q+n-\frac{1}{2}}, \quad q \in \{p, p+1\} \right\}$$

is obviously at least $(\frac{1}{2\sqrt{\lambda}})^{n-1}$. For such (x_2, \dots, x_n) , we let $\max_{x_1} \left| \frac{\partial^p u}{\partial x_1^p} \right| = \left| \frac{\partial^p u}{\partial x_1^p}(x_1^0) \right|$ with $x_1^0 \in [0, \frac{1}{2\sqrt{\lambda}}]$. For any $x_1 \in [0, \frac{1}{2\sqrt{\lambda}}]$, we have

$$\left| \frac{\partial^p u}{\partial x_1^p}(x_1^0) \right|^2 = \left| \frac{\partial^p u}{\partial x_1^p}(x_1) - \int_{x_1^0}^{x_1} \frac{\partial^{p+1} u}{\partial x_1^{p+1}}(\tau) d\tau \right|^2.$$

Integrating both sides of the equality with respect to x_1 and using Jensen's inequality, we obtain

$$\begin{aligned} & \frac{1}{2\sqrt{\lambda}} \left(\max_{x_1} \left| \frac{\partial^p u}{\partial x_1^p} \right| \right)^2 \\ &= \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \frac{\partial^p u}{\partial x_1^p} - \int_{x_1^0}^{x_1} \frac{\partial^{p+1} u}{\partial x_1^{p+1}}(\tau) d\tau \right|^2 dx_1 \\ &= \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \frac{\partial^p u}{\partial x_1^p} \right|^2 dx_1 + \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \int_{x_1^0}^{x_1} \frac{\partial^{p+1} u}{\partial x_1^{p+1}}(\tau) d\tau \right|^2 dx_1 \\ &\quad - 2\operatorname{Re} \int_0^{\frac{1}{2\sqrt{\lambda}}} \left[\frac{\partial^p u}{\partial x_1^p} \overline{\int_{x_1^0}^{x_1} \frac{\partial^{p+1} u}{\partial x_1^{p+1}}(\tau) d\tau} \right] dx_1 \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \frac{\partial^p u}{\partial x_1^p} \right|^2 dx_1 + \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \int_{x_1^0}^{x_1} \frac{\partial^{p+1} u}{\partial x_1^{p+1}}(\tau) d\tau \right|^2 dx_1 \\
 &\quad + \frac{1}{2} \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \frac{\partial^p u}{\partial x_1^p} \right|^2 dx_1 + 2 \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \int_{x_1^0}^{x_1} \frac{\partial^{p+1} u}{\partial x_1^{p+1}}(\tau) d\tau \right|^2 dx_1 \\
 &= \frac{3}{2} \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \frac{\partial^p u}{\partial x_1^p} \right|^2 dx_1 + 3 \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \int_{x_1^0}^{x_1} \frac{\partial^{p+1} u}{\partial x_1^{p+1}}(\tau) d\tau \right|^2 dx_1 \\
 &\leq \frac{3}{2} \int_0^{\frac{1}{2\sqrt{\lambda}}} \left| \frac{\partial^p u}{\partial x_1^p} \right|^2 dx_1 + 3 \int_0^{\frac{1}{2\sqrt{\lambda}}} (x_1 - x_1^0) \int_{x_1^0}^{x_1} \left| \frac{\partial^{p+1} u}{\partial x_1^{p+1}}(\tau) \right|^2 d\tau dx_1 \\
 &\leq \frac{3}{2} \left\| \frac{\partial^p u}{\partial x_1^p} \right\|_{L^2([0, \frac{1}{2\sqrt{\lambda}}])}^2 + 3 \left\| \frac{\partial^{p+1} u}{\partial x_1^{p+1}} \right\|_{L^2([0, \frac{1}{2\sqrt{\lambda}}])}^2 \int_0^{\frac{1}{2\sqrt{\lambda}}} |x_1 - x_1^0| dx_1 \\
 &\leq \frac{3}{2} \left[\left\| \frac{\partial^p u}{\partial x_1^p} \right\|_{L^2([0, \frac{1}{2\sqrt{\lambda}}])}^2 + \frac{1}{4\lambda} \left\| \frac{\partial^{p+1} u}{\partial x_1^{p+1}} \right\|_{L^2([0, \frac{1}{2\sqrt{\lambda}}])}^2 \right].
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \max_{x_1} \left| \frac{\partial^p u}{\partial x_1^p} \right| &\leq \sqrt{\frac{3}{2} \left(2\sqrt{\lambda} \left\| \frac{\partial^p u}{\partial x_1^p} \right\|_{L^2([0, \frac{1}{2\sqrt{\lambda}}])}^2 + \frac{1}{2\sqrt{\lambda}} \left\| \frac{\partial^{p+1} u}{\partial x_1^{p+1}} \right\|_{L^2([0, \frac{1}{2\sqrt{\lambda}}])}^2 \right)} \\
 &= \sqrt{3 \left(\sqrt{\lambda} \left\| \frac{\partial^p u}{\partial x_1^p} \right\|_{L^2([0, \frac{1}{2\sqrt{\lambda}}])}^2 + \frac{1}{4\sqrt{\lambda}} \left\| \frac{\partial^{p+1} u}{\partial x_1^{p+1}} \right\|_{L^2([0, \frac{1}{2\sqrt{\lambda}}])}^2 \right)} \\
 (2.18) \quad &\leq \sqrt{3 \left(\sqrt{\lambda} \cdot 2\beta_p^2 \lambda^{p+n-\frac{1}{2}} + \frac{1}{4\sqrt{\lambda}} \cdot 2\beta_{p+1}^2 \lambda^{p+1+n-\frac{1}{2}} \right)} \\
 &= \sqrt{3} \cdot \lambda^{\frac{p+n}{2}} \sqrt{2\beta_p^2 + \frac{1}{2}\beta_{p+1}^2} \\
 &\leq \sqrt{6} \lambda^{\frac{p+n}{2}} \sqrt{\beta_p^2 + \beta_{p+1}^2}.
 \end{aligned}$$

Let $u(x_1) = v_1(x_1) + \sqrt{-1} v_2(x_1)$. We consider separately the following two cases:

Case 1: If $\max|u| \geq 6$, then at least one of $\max|v_1|$ and $\max|v_2|$ is greater than or equal to 3. Without loss of generality, we assume that $\max|v_1| \geq 3$ and apply Lemma 2.3 to the function v_1 . If v_1 satisfies

$$\max|v_1'| < 4^{p+1} \lambda^{\frac{1}{2}} \max|v_1|,$$

then there exists a subinterval $[t_1, t_2]$ of the length $2^{-2p-3} \lambda^{-\frac{1}{2}}$ on which $|v_1| \geq \frac{1}{2} \max|v_1| \geq \frac{3}{2}$. In fact, we can choose two points t_1, t_2 of the interval $[0, \frac{1}{2\sqrt{\lambda}}]$ such that $|v_1(t_2)| = \max|v_1|$, $|v_1(t_1)| = \frac{1}{2} \max|v_1|$ and $|v_1| \geq \frac{1}{2} \max|v_1|$ on $[t_1, t_2]$. By the mean value theorem,

we get

$$\begin{aligned}
\frac{1}{2} \max |v_1| &\leq |v_1(t_2) - v_1(t_1)| \\
&= |v_1'(\xi)| \cdot |t_2 - t_1| \\
&\leq \max |v_1'| \cdot |t_2 - t_1| \\
&\leq 4^{p+1} \lambda^{\frac{1}{2}} \max |v_1| \cdot |t_2 - t_1|.
\end{aligned}$$

Hence,

$$|t_2 - t_1| \geq 2^{-2p-3} \lambda^{-\frac{1}{2}}.$$

Since $|u(x_1) - e^{\sqrt{-1}\xi_1 x_1}| \geq |u(x_1)| - 1 \geq |v_1(x_1)| - 1 \geq \frac{1}{2}$ on the subinterval $[t_1, t_2]$,

$$(2.19) \quad \int_0^{\frac{1}{2\sqrt{\lambda}}} |u(x_1) - e^{\sqrt{-1}\xi_1 x_1}|^2 dx_1 \geq 2^{-2p-5} \lambda^{-\frac{1}{2}}.$$

On the other hand, if v_1 satisfies

$$\max |v_1'| \leq 2^{p-1} \left(\frac{\max |v_1^{(p)}|}{\max |v_1|} \right)^{\frac{1}{p}} \max |v_1|,$$

then it follows from the mean value theorem and (2.18) that the length of the subinterval of $[0, \frac{1}{2\sqrt{\lambda}}]$ on which $|v_1| \geq \frac{3}{2}$, is at least $2^{-p} 3^{\frac{1}{p}} 6^{-\frac{1}{2p}} (\beta_p^2 + \beta_{p+1}^2)^{-\frac{1}{2p}} \lambda^{-\frac{1}{2} - \frac{n}{2p}}$, which yields

$$(2.20) \quad \int_0^{\frac{1}{2\sqrt{\lambda}}} |u(x_1) - e^{\sqrt{-1}\xi_1 x_1}|^2 dx_1 \geq 2^{-p-2} 3^{\frac{1}{p}} 6^{-\frac{1}{2p}} (\beta_p^2 + \beta_{p+1}^2)^{-\frac{1}{2p}} \lambda^{-\frac{1}{2} - \frac{n}{2p}}.$$

Case 2: Assume now that $\max |u| < 6$. The latter means that $\max |v_1| < 6$ and $\max |v_2| < 6$. Since $v_1(0) = v_2(0) = 0$, there exists a subinterval I of $[0, \frac{1}{2\sqrt{\lambda}}]$ such that $|v_1| \leq \frac{1}{3}$, $|v_2| \leq \frac{1}{3}$ on I , which implies

$$|u(x_1) - e^{\sqrt{-1}\xi_1 x_1}|^2 \geq (|u(x_1)| - 1)^2 \geq \left(1 - \frac{\sqrt{2}}{3}\right)^2 \geq \frac{1}{4}.$$

Next we apply Lemma 2.3 to the functions v_1 and v_2 . We find out that the length of I is at least

$$\min \left\{ 3^{-2} 2^{-2p-3} \lambda^{-\frac{1}{2}}, \quad 3^{-2} 2^{-p} 6^{\frac{1}{2p}} (\beta_p^2 + \beta_{p+1}^2)^{-\frac{1}{2p}} \lambda^{-\frac{1}{2} - \frac{n}{2p}} \right\}.$$

This implies

$$\int_0^{\frac{1}{2\sqrt{\lambda}}} |u(x_1) - e^{\sqrt{-1}\xi_1 x_1}|^2 dx_1 \geq \frac{\lambda^{-\frac{1}{2}}}{9} \min \left\{ 2^{-2p-5}, \quad 2^{-p-2} 6^{\frac{1}{2p}} (\beta_p^2 + \beta_{p+1}^2)^{-\frac{1}{2p}} \lambda^{-\frac{1}{2} - \frac{n}{2p}} \right\}.$$

This completes the proof of the lemma. \square

Proof of Theorem 1: According to Lemma 2.4, for each l_i and p , we have

$$\left\| u - e^{\sqrt{-1}\langle \xi, x \rangle} \right\|_{L^2(T_{l_j})}^2 \geq \frac{1}{9 \cdot 2^{n-1}} \min \left\{ 2^{-2p-5} \lambda^{-\frac{n}{2}}, \quad 2^{-p-2} 6^{\frac{1}{2p}} (\beta_p^2 + \beta_{p+1}^2)^{-\frac{1}{2p}} \lambda^{-\frac{n}{2} - \frac{n}{2p}} \right\},$$

where

$$\beta_{p+1}^2 = \left(\frac{n+2}{4n\pi^2} \right)^{\frac{n}{2}} B_n V^2 D_p.$$

Next we estimate the sequence D_p . A direct inspection shows that

$$D_p < 2^{(2n+18)p^2}.$$

This implies that

$$(\beta_p^2 + \beta_{p+1}^2)^{-\frac{1}{2p}} \geq (2\beta_{p+1}^2)^{-\frac{1}{2p}} > 2^{-\frac{1}{2p} - (n+9)p} \left(\frac{4n\pi^2}{n+2} \right)^{\frac{n}{4p}} B_n^{-\frac{1}{2p}} V^{-\frac{1}{p}}.$$

Hence

$$\left\| u - e^{\sqrt{-1}\langle \xi, x \rangle} \right\|_{L^2(T_{l_i})}^2 \geq \frac{1}{9 \cdot 2^{n-1}} \min \left\{ 2^{-2p-5} \lambda^{-\frac{n}{2}}, 2^{-2-(n+10)p} \left(\frac{V\lambda^{\frac{n}{2}}}{\alpha_1} \right)^{-\frac{1}{p}} \lambda^{-\frac{n}{2}} \right\},$$

where

$$\alpha_1 = \sqrt{\frac{3}{B_n} \left(\frac{4n\pi^2}{n+2} \right)^{\frac{n}{2}}}.$$

Now we choose

$$p = \left\lceil \sqrt{\frac{\log_2[(V/\alpha_1)^{n-1} \lambda^{\frac{n}{2}}]}{n+12}} \right\rceil,$$

then

$$2^{-2-(n+10)p} \geq \left(\frac{V}{\alpha_1} \right)^{-\frac{n-2}{2p}} \left(\frac{V\lambda}{\alpha_1} \right)^{-\frac{n}{2p}}.$$

Therefore, we obtain

$$(2.21) \quad \left\| u - e^{\sqrt{-1}\langle \xi, x \rangle} \right\|_{L^2(T_{l_i})}^2 \geq \frac{1}{9 \cdot 2^{n-1}} \left\{ \left(\frac{V}{\alpha_1} \right)^{\frac{n}{2}} \left(\frac{V\lambda}{\alpha_1} \right)^{-\frac{n}{2} - \frac{n}{p}} \right\}.$$

Notice that for each i the number of these n -dimensional rectangles T_{l_i} is at least

$$N_i = \left\lceil \frac{A_i}{6} \lambda^{\frac{n-1}{2}} \right\rceil.$$

Summing the inequality (2.21) for all $l_i = 1, \dots, N_i$ and all $i = 1, \dots, m$, we immediately get

$$\begin{aligned} V - (2\pi)^n F_\lambda(\xi) &= \left\| u - e^{\sqrt{-1}\langle \xi, x \rangle} \right\|_{L^2(\Omega)}^2 \\ &\geq \frac{1}{9 \cdot 2^{n-1}} \left\{ \left(\frac{V}{\alpha_1} \right)^{\frac{n}{2}} \left(\frac{V\lambda}{\alpha_1} \right)^{-\frac{n}{2} - \frac{n}{p}} \right\} \sum_{i=1}^m \left[\frac{A_i}{6} \lambda^{\frac{n-1}{2}} \right] \Theta(\lambda - \lambda_0) \\ &\geq \frac{\alpha_1^{-\frac{1}{2}} V^{\frac{1}{2}}}{9^2 \cdot 2^n} \left(\frac{V\lambda}{\alpha_1} \right)^{-\frac{1}{2} - \frac{n}{p}} A(\partial\Omega) \Theta(\lambda - \lambda_0), \end{aligned}$$

where

$$\lambda_0 = \max \left\{ \frac{4n}{\min_i \{d_i^2\}}, \left(\frac{\alpha_1}{V} \right)^{\frac{2}{n}}, 2^{\frac{2(n+12)}{n}} \left(\frac{\alpha_1}{V} \right)^{\frac{2(n-1)}{n}}, \left(\frac{12}{\min_i \{A_i\}} \right)^{\frac{2}{n-1}} \right\}.$$

This yields the following upper bound on F_λ :

$$(2.22) \quad F_\lambda(\xi) \leq \frac{V}{(2\pi)^n} \left[1 - \alpha_2 V^{-\frac{1}{2}} A(\partial\Omega) \left(\frac{V\lambda}{\alpha_1} \right)^{-\frac{1}{2}-\frac{n}{p}} \Theta(\lambda - \lambda_0) \right] \triangleq M(\lambda),$$

where

$$\alpha_2 = \frac{\alpha_1^{-\frac{1}{2}}}{9^2 \cdot 2^n}.$$

By Lemma 2.1, we obtain

$$\begin{aligned} \sum_{j=1}^{N(\lambda)} \lambda_j &= \int_{\Omega} |\xi|^2 F_\lambda(\xi) d\xi \\ &\geq \frac{n}{n+2} B_n^{-\frac{2}{n}} N(\lambda)^{\frac{n+2}{n}} M(\lambda)^{-\frac{2}{n}} \\ &= \frac{n}{n+2} B_n^{-\frac{2}{n}} N(\lambda)^{\frac{n+2}{n}} (2\pi)^2 V^{-\frac{2}{n}} \left[1 - \alpha_2 V^{-\frac{1}{2}} A(\partial\Omega) \left(\frac{V\lambda}{\alpha_1} \right)^{-\frac{1}{2}-\frac{n}{p}} \Theta(\lambda - \lambda_0) \right]^{-\frac{2}{n}} \\ &\geq \frac{n}{n+2} B_n^{-\frac{2}{n}} N(\lambda)^{\frac{n+2}{n}} (2\pi)^2 V^{-\frac{2}{n}} \left[1 + \frac{2\alpha_2}{n} V^{-\frac{1}{2}} A(\partial\Omega) \left(\frac{V\lambda}{\alpha_1} \right)^{-\frac{1}{2}-\frac{n}{p}} \Theta(\lambda - \lambda_0) \right] \\ &= \frac{n}{n+2} \frac{(2\pi)^2}{(B_n V)^{\frac{2}{n}}} N(\lambda)^{\frac{n+2}{n}} + \frac{2\alpha_2}{n+2} \frac{(2\pi)^2}{(B_n V)^{\frac{2}{n}}} N(\lambda)^{\frac{n+2}{n}} V^{-\frac{1}{2}} A(\partial\Omega) \left(\frac{V\lambda}{\alpha_1} \right)^{-\frac{1}{2}-\frac{n}{p}} \Theta(\lambda - \lambda_0). \end{aligned}$$

Taking k large enough such that $\lambda = \lambda_k$, we immediately get

$$\frac{1}{k} \sum_{j=1}^k \lambda_j \geq \frac{n}{n+2} \frac{(2\pi)^2}{(B_n V)^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{8\pi^2}{9^2 2^n (n+2) B_n^{\frac{2}{n}}} \frac{A(\partial\Omega)}{V^{1+\frac{2}{n}}} k^{\frac{2}{n}} \lambda_k^{-\frac{1}{2}} \left(\frac{V\lambda_k}{\alpha_1} \right)^{-n\varepsilon(k)} \Theta(\lambda_k - \lambda_0),$$

where

$$\varepsilon(k) = \frac{1}{p} = \left[\sqrt{\frac{\log_2 [(V/\alpha_1)^{n-1} \lambda_k^{\frac{n}{2}}]}{n+12}} \right]^{-1}. \quad \square$$

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